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Nonabelian Monopoles and the Vortices that Confine Them

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Abstract:

Nonabelian magnetic monopoles of Goddard-Nuyts-Olive-Weinberg type have recently been shown to appear as the dominant infrared degrees of freedom in a class of softly broken $\mathcal{N} = 2$ supersymmetric gauge theories in which the gauge group G is broken to various nonabelian subgroups H by an adjoint Higgs VEV. When the low-energy gauge group H is further broken completely by e.g. squark VEVs, the monopoles representing $\pi_2(G/H)$ are confined by the nonabelian vortices arising from the breaking of H, discussed recently (hep-th/0307278). By considering the system with $G = SU(N+1)$, $H = \frac{SU(N)\times U(1)}{\mathbb{Z}_N}$, as an example, we show that the total magnetic flux of the minimal monopole agrees precisely with the total magnetic flux flowing along the single minimal vortex. The possibility for such an analysis reflects the presence of free parameters in the theory - the bare quark mass m and the adjoint mass μ - such that for $m \gg \mu$ the topologically nontrivial solutions of vortices and of unconfined monopoles exist at distinct energy scales.

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1 Introduction

Nonabelian monopoles in spontaneously broken gauge theories have remained somewhat obscure objects for a long time in spite of many investigations [1]-[10]. Apart from the often discussed applications in conformally invariant $\mathcal{N} = 4$ theories, few field theory models were known where such objects play an important dynamical role. Although many $\mathcal{N} = 1$ gauge theories, such as SQCD with appropriate numbers of flavors, are believed to possess Seiberg duals [11], the origin of the "dual quarks" appearing in these models remains mysterious.

A series of papers on softly broken $\mathcal{N} = 2$ gauge theories with gauge groups $SU(N)$, $USp(2N)$ and $SO(N)$ and various numbers of flavors of fundamental matter have, however, changed the situation [12, 13, 14]. In particular, it was pointed out [15] that the "dual quarks" appearing as the low-energy degrees of freedom of the $G = SU(N)$, $USp(2N)$ or $SO(N)$ theory, which carry the nonabelian $SU(r) \subset G$ charges, can be identified with the "semiclassical" nonabelian monopoles studied earlier by Goddard, Nuyts, Olive [4] and by E. Weinberg [7]. Also, all of the confining vacua in strongly coupled $USp(2N)$ and $SO(N)$ theories with flavors and with zero bare quark masses, involve these objects in a deformed SCFT.

Very recently, with A. Yung, we have proven the existence of nonabelian *vortices* in the same class of models [16]. The analysis was done semiclassicaly, in the region of large bare quark masses (and so large adjoint scalar VEVS), but the presence of an appropriate number of fermions makes the results quantum mechanically correct. In particular, a continuous family of degenerate vortex solution have been constructed, showing the truely nonabelian nature of these vortices [1].

In this paper, we discuss some aspects *relating* nonabelian vortices and monopoles appearing in the softly broken $\mathcal{N} = 2$ $G = SU(N)$ theories with N_f flavors. The gauge group is broken at two very different mass scales, $v_1 \gg v_2$,

$$G \xrightarrow{v_1} H \xrightarrow{v_2} 0. \tag{1.1}$$

[1]Deceptively similar, though different, vortex configurations have been studied independently by Hanany and Tong [17, 18].

For concreteness we shall study the case of the symmetry breaking $G = SU(N+1)$, $H = \frac{SU(N)\times U(1)}{\mathbb{Z}_N}$: the symmetry breaking at the higher mass scale is due to the adjoint scalar VEV which is proportional to the bare quark masses $v_1 \sim m$ (see Eq.(2.9) below); the squark VEVS break H at much smaller mass scale, $v_2 \sim \sqrt{\mu m}$, where μ is the small adjoint scalar mass, breaking the supersymmetry to $\mathcal{N} = 1$. The model is basically the bosonic sector of the $\mathcal{N} = 2$ supersymmetric gauge theories [19]-[21]. The full supersymmetric dynamics of the theory involving fermions is however needed to show the quantum mechanical stability of what is found here semiclassically [15, 16].

Strictly speaking, neither the monopoles nor vortices exist in this theory as static, topologically stable configurations, as $\pi_2(G) = \pi_1(G) = 0$. Nonetheless, the existence of the different scales in the theory allows us to study these configurations as approximate, topologically stable static configurations of effective theories defined at different scales. At high energies where the effects of the smaller condensates are negligible, the theory possesses the nonabelian monopoles representing nontrivial elements of $\pi_2(G/H)$ and transforming as multiplets of the dual gauge group $\tilde{H}$. At lower energies the light fields are described by an effective H theory in the Higgs phase. This theory possesses vortices of $\pi_1(H)$ which are stable in so far as the pair production of the massive monopoles is suppressed.

The equivalence of the homotopy groups $\pi_2(G/H) \sim \pi_1(H)$ implies that the monopoles are confined. Although the configuration of an isolated monopole has an infinite energy in the Higgs phase ($v_2 \neq 0$), the flux of the monopole can be whisked away by a single vortex, so that a monopole-vortex-antimonopole configuration has a finite energy. *In making this discussion more quantitative, we show that the flux through a small sphere around a monopole exactly matches the flux along the vortex through a plane perpendicular to it, far from the monopole.*

Although this result is in a sense to be expected, it is actually a quite nontrivial matter to show it, as in the present model the monopoles are "made of" gauge bosons and adjoint scalars while the vortices are nontrivial configurations involving gauge fields and squarks fields only; the two types of configurations appear as solutions of two different effective theories valid at different energy scales.

More importantly, this discussion shows that the monopoles indeed form a nonabelian (dual) gauge multiplet, since continuous transformations of the vortex solutions have recently been explicitly constructed [16], proving their nonabelian nature.

2 Nonabelian Monopoles and Vortices in $SU(N)$ Gauge Theories

2.1 High-energy theory: BPS monopoles

We start our discussion by considering the monopoles and vortices arising in a system with symmetry breaking $SU(N+1) \to SU(N) \times U(1)$. The field theory considered here is essentially the bosonic sector of $\mathcal{N}=2$ supersymmetric $SU(N+1)$ gauge theory [21]. The discussion of this section is semi-classical, although when embedded in the $\mathcal{N}=2$ theory and with appropriate number of flavors (in this case, $2N \le N_f \le 2N+2$), the whole discussion is valid quantum mechanically.

For concreteness, in this subsection we discuss the $\mathcal{N}=2$, $SU(3)$ gauge theory with $n_f = 4,5$ flavors of hypermultiplets ("quarks"). The generalization to systems with more general pattern of symmetry breaking $SU(N+1) \to SU(r) \times U(1)^{N-r+1}$ is straightforward. The results for the monopole-vortex flux matching in the next section will be given for $SU(N+1) \to SU(N) \times U(1)$ cases as well.

The Lagrangian of this theory has the structure

$$\mathcal{L} = \frac{1}{8\pi} \mathrm{Im}\, S_{cl} \left[\int d^4\theta \, \Phi^\dagger e^V \Phi + \int d^2\theta \, \frac{1}{2} W W \right] + \mathcal{L}^{(quarks)} + \int d^2\theta \, \mu \, \mathrm{Tr}\Phi^2; \tag{2.1}$$

$$\mathcal{L}^{(quarks)} = \sum_i [\int d^4\theta \, \{ Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger \} + \int d^2\theta \, \{ \sqrt{2} \tilde{Q}_i \Phi Q^i + m \tilde{Q}_i Q^i \} \tag{2.2}$$

where m is the bare mass of the quarks and we have defined the complex coupling constant

$$S_{cl} \equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}. \tag{2.3}$$

The parameter μ is the mass of the adjoint chiral multiplet, which breaks the supersymmetry to $\mathcal{N}=1$.

In order to discuss unconfined monopoles, however, we must set $\mu = 0$ (see Subsec. 3 below) and so preserve the full $\mathcal{N}=2$ supersymmetry. After eliminating the auxiliary fields the bosonic Lagrangian becomes

$$\mathcal{L} = \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2} |\mathcal{D}_\mu \phi|^2 + |\mathcal{D}_\mu Q|^2 + \left| \mathcal{D}_\mu \bar{\tilde{Q}} \right|^2 + \mathcal{L}_1 + \mathcal{L}_2, \tag{2.4}$$

where

$$\mathcal{L}_1 \;\; = \;\; -\frac{1}{8} \sum_A [\frac{1}{g^2} (-i) f_{ABC} \, \phi_B^\dagger \phi_C + Q^\dagger t^A Q - \tilde{Q} t^A \tilde{Q}^\dagger]^2$$

$$= \quad -\frac{1}{8}\sum_A \left(t^A_{ij}\,[\frac{1}{g^2}(-2)\,[\phi^\dagger,\phi]_{ji} + Q^\dagger_j Q_i - \tilde{Q}_j \tilde{Q}^\dagger_i]^2 \right)^2 ; \tag{2.5}$$

$$\begin{aligned} \mathcal{L}_2 \quad &= \quad -g^2 |\mu\,\phi^A + \sqrt{2}\,\tilde{Q}\,t^A Q|^2 - \tilde{Q}\,[m+\sqrt{2}\phi]\,[m+\sqrt{2}\phi]^\dagger\,\tilde{Q}^\dagger \\ &- \quad Q^\dagger\,[m+\sqrt{2}\phi]^\dagger\,[m+\sqrt{2}\phi]\,Q. \end{aligned} \tag{2.6}$$

In the construction of the monopole solutions we shall consider only the VEVs and fluctuations around them which satisfy

$$[\phi^\dagger,\phi] = 0, \qquad Q_i = \tilde{Q}^\dagger_i, \tag{2.7}$$

and hence $\mathcal{L}_1$, $\mathcal{L}_2$ can be set identically to zero.

This theory has a number of vacua parametrized by the integer r, which is the rank of the unbroken nonabelian gauge symmetry plus one [12, 13]. For concreteness we first consider the $r = 2$ vacuum, in which the adjoint scalar has a nonvanishing VEV ($\Phi = t^a \phi^a$)

$$diag.\langle\Phi\rangle = v_1\,(1,1,-2); \quad \langle\phi^b\rangle = 0, \quad b = 1,2,3, \quad \langle\phi^8\rangle = -2\sqrt{3}\,v_1, \tag{2.8}$$

while the squark VEVs are set to zero, $Q_i = \tilde{Q}^\dagger_i = 0$. We will consider the semiclassical regime in which the bare quark mass is much larger than the QCD scale, in which case $v_1 = m/\sqrt{2}$ and so

$$diag.\langle\Phi\rangle = \frac{1}{\sqrt{2}}(m, m, -2\,m); \quad \langle\phi^8\rangle = -\sqrt{6}m. \tag{2.9}$$

This VEV breaks the gauge symmetry as

$$SU(3) \to \frac{SU(2)\times U(1)}{\mathbb{Z}_2}, \tag{2.10}$$

where the $\mathbb{Z}_2$ factor arises because $SU(2)$ and $U(1)$ share the common element $-\mathbf{1}$.

The nontrivial homotopy groups

$$\pi_2(\frac{SU(3)}{SU(2)\times U(1)/\mathbb{Z}_2}) = \pi_1(SU(2)\times U(1)/\mathbb{Z}_2) = \mathbb{Z} \tag{2.11}$$

imply that nontrivial monopole solutions exist. The energy of such configurations may be read from the Hamiltonian

$$H = \int d^3x \left[\frac{1}{4g^2}(F^A_{ij})^2 + \frac{1}{g^2}|\mathcal{D}_i\phi^A|^2\right] = \int d^3x \left[\frac{1}{4g^2}(F^A_{ij})^2 + \frac{1}{2g^2}|\mathcal{D}_i\phi^A|^2\right] \tag{2.12}$$

where in the second formula we have kept only the real part of ϕ^A. Note that we have restricted our interest to static configurations with no electric flux. For real ϕ^A, $f_{ABC}\,\phi_B^\dagger\phi_C = 0$ so neither $\mathcal{L}_1$ nor $\mathcal{L}_2$ contribute. Rewriting the Hamiltonian as

$$H = \int d^3x\,\Big[\frac{1}{4g^2}|F_{ij}^A \pm \epsilon_{ijk}(\mathcal{D}_k\phi)^A|^2 \pm \frac{1}{2}\partial_k(\epsilon_{ijk}F_{ij}^A\phi^A)\Big] \tag{2.13}$$

it becomes clearer that BPS monopole configurations must satisfy the nonabelian Bogomolny equations

$$B_k^A = -(\mathcal{D}_k\phi)^A; \qquad B_k^A = \frac{1}{2}\,\epsilon_{ijk}F_{ij}^A. \tag{2.14}$$

The BPS bound on the monopole mass is then (see Eq.(4.4), Eq.(4.5), Eq.(4.6) below)

$$H = \int dS\cdot(\phi^A\mathbf{B}^A) = \frac{2\pi}{g}\,3\,v_1\,m, \qquad m = 1,2,\ldots. \tag{2.15}$$

2.2 Low-energy theory: Vortices

Vortices appear in the low-energy theory when the symmetry group $\frac{SU(2)\times U(1)}{\mathbb{Z}_2}$ is further spontaneously broken by squark VEVs [16]. Upon turning on an adjoint mass perturbation ($\mu \neq 0$), the squark VEVs take a color-flavor diagonal form ($\xi \equiv \mu\,m$):

$$< q^{kA} > = < \bar{\tilde{q}}^{kA} > = \sqrt{\frac{\xi}{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = v_2\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{2.16}$$

where only the first two color and flavor components are explicitly shown (all other components being identically zero in the vortex solution). The light fields enter the $SU(2)\times U(1)$ Lagrangian at scales between v_1 and v_2 (we set $\mathcal{L}_1 = 0$) as

$$\begin{aligned}\mathcal{L} &= \frac{1}{4g_2^2}(F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2}(F_{\mu\nu}^0)^2 + \frac{1}{g_2^2}|\mathcal{D}_\mu\phi^a|^2 + \frac{1}{g_1^2}|\mathcal{D}_\mu\phi^0|^2 + |\mathcal{D}_\mu Q|^2 + \Big|\mathcal{D}_\mu\bar{\tilde{Q}}\Big|^2 \\ &- g_2^2|\,\mu\,\phi^8 + \sqrt{2}\,\tilde{Q}\,t^8 Q\,|^2 - g_1^2|\sqrt{2}\,\tilde{Q}\,t^a Q\,|^2 - \tilde{Q}\,[m+\sqrt{2}\phi]\,[m+\sqrt{2}\phi]^\dagger\,\tilde{Q}^\dagger \\ &- Q^\dagger\,[m+\sqrt{2}\phi]^\dagger\,[m+\sqrt{2}\phi]\,Q, \end{aligned} \tag{2.17}$$

where $a = 1,2,3$ labels the $SU(2)$ generators, $t^a = S^a$; the index 0 refers to $t^8 = \frac{1}{2\sqrt{3}}\,\mathrm{diag}(1,1,-2)$. We have taken into account the different renormalization effects in the $SU(2)$ sector and $U(1)$ sector and distinguished the coupling constants g_2 (of $SU(2)$ interactions) and g_1 (of $U(1)$).

Note that the model discussed by Hanany and Tong [17, 18] is different, as the FI term in the $U(1)$ part is put in by hand, while in our model the corresponding term is an F term, arising naturally from the $SU(3) \to SU(2) \times U(1)$ breaking. Also, our monopoles and vortices have quantum mechanical meaning as the $SU(2) \times U(1)$ is infrared free in the scales between v_1 and v_2. While they have found two vortices ending on each monopole in their model, we will see that the monopoles of our model are each confined by a single vortex.

The static field energy of an arbitrary configuration without electric flux is

$$\begin{aligned} H &= \int d^3x \Big[\frac{1}{4g_2^2}(F_{ij}^a)^2 + \frac{1}{4g_1^2}(F_{ij}^0)^2 + \frac{1}{g_2^2}|\mathcal{D}_i\phi^a|^2 + \frac{1}{g_1^2}|\mathcal{D}_i\phi^0|^2 + |\mathcal{D}_i Q|^2 + \Big|\mathcal{D}_i\bar{\tilde{Q}}\Big|^2 + \\ &+ g_2^2|\, \mu\, \phi^8 + \sqrt{2}\, \tilde{Q}\, t^8 Q\, |^2 + g_1^2|\sqrt{2}\, \tilde{Q}\, t^a Q\, |^2 + \tilde{Q}\, [m + \sqrt{2}\phi]\, [m + \sqrt{2}\phi]^\dagger\, \tilde{Q}^\dagger \\ &+ Q^\dagger\, [m + \sqrt{2}\phi]^\dagger\, [m + \sqrt{2}\phi]\, Q \Big]. \end{aligned} \tag{2.18}$$

Now let us retrict our attention to those configurations in which the adjoint scalar is fixed to its VEV,

$$\phi = v_1\, t^8, \tag{2.19}$$

which is constant and commutes with t^a and t^8 and also satisfies $\mathcal{D}_i\phi^a \to 0$. By also keeping $Q = \tilde{Q}^\dagger$, rescaling $Q = \frac{1}{\sqrt{2}}q$, and keeping the first two color and flavor components of these to be nonvanishing, one obtains the Hamiltonian

$$\begin{aligned} H &= \int d^3x \Big[|\frac{1}{2g_1} F_{ij}^0 \pm \epsilon_{ij} g_1(-\sqrt{3}\, m\, \mu\, + \frac{1}{4\sqrt{3}}\, q^\dagger\, q\,)|^2 + \\ &+ |\frac{1}{2g_2} F_{ij}^a \pm \epsilon_{ij} \frac{g_2}{4}\, q^\dagger\, S^a q\, |^2 + \frac{1}{2}\left|\mathcal{D}_i\, q^A \pm i\epsilon_{ij}\mathcal{D}_j\, q^A\right|^2 \pm 2\sqrt{3}\, m\, \mu\, \tilde{F}^{(0)} \Big] \end{aligned} \tag{2.20}$$

where $\tilde{F}^{(0)} \equiv \frac{1}{2}\epsilon_{ij} F_{ij}^0$ is the $U(1)$ flux. This way one finds the nonabelian Bogomolny equations ($\varepsilon = \pm 1$),

$$\begin{aligned} &\frac{1}{2g_2} F_{ij}^{(a)} + \frac{g_2}{4}\, \varepsilon\, \left(\bar{q}_A S^a q^A\right) \epsilon_{ij} = 0, \qquad a = 1, 2, 3; \\ &\frac{1}{2g_1} F_{ij}^{(0)} + \frac{g_1}{4\sqrt{3}}\, \varepsilon\, \left(|q^A|^2 - 2\xi\right) \epsilon_{ij} = 0; \\ &\nabla_i\, q^A + i\, \varepsilon\, \epsilon_{ij}\, \nabla_j\, q^A = 0, \qquad A = 1, 2, \dots, N_f. \end{aligned} \tag{2.21}$$

The properties of the BPS vortex solutions have been discussed in detail recently [16]. In fact, there exists a continuously degenerate family of vortex solutions of Eq.(2.21),

parametrized by $SU(2)_{C+F}/U(1) = CP^1 = S^2$. This is due to the system's exact symmetry $SU(2)_{C+F} \subset SU(3)_c \times SU(n_f)_F$ (remember $n_f = 4, 5$) which is broken only by individual vortex configurations. In [16] it was also verified that such an exact symmetry is not spontaneously broken. In other words, the dual of the original $SU(2) \times U(1)$ theory in Higgs phase is indeed a confining $\mathcal{N} = 1$ $SU(2)$ theory, with two vacua!

This implies the existence of the corresponding degenerate family of monopoles which appear as sources of these vortices. For consistency, the monopole and vortex fluxes must match precisely, a fact to be proven in Section 4 below.

3 Monopoles and Vortices Are Incompatible

It might be tempting at this point to try to search for a static solution of the non-abelian Bogomolny equations containing both the vortex and the monopole. However no such solution exists. Monopoles are topologically stable only if $\mu = 0$: they represent $\pi_2(SU(3)/(SU(2) \times U(1)/\mathbb{Z}_2)) \sim \pi_1(SU(2) \times U(1)/\mathbb{Z}_2)$. At the low energy scales in which our $(SU(2) \times U(1))/\mathbb{Z}_2$ symmetry is entirely broken by the bare adjoint chiral multiplet mass, topologically nontrivial monopole configurations are classified by the homotopy group $\pi_2(SU(3)) = 0$ and thus the topological stability of our monopoles fails.

On the other hand vortices exist in the Higgs phase of the H theory (which requires $\mu \neq 0$): they represent the fundamental group $\pi_1(SU(2) \times U(1)/\mathbb{Z}_2)$,[2] but the vortices are stable only approximately in the theory defined at scales much lower than v_1, where monopole production is suppressed by a tiny barrier penetration factor [22].

Mathematically, the nonexistence of the vortex solution in the high-energy $SU(3)$ theory reflects the fact that it is simply connected ($\pi_1(SU(3)) = \mathbf{0}$). Physically, any vortex can be attached to a monopole and antimonopole at the two ends: clearly the monopole-vortex-antimonopole configuration cannot be a configuration of minimum energy, as the energy decreases as the vortex becomes shorter.

Summarizing, the monopoles and vortices are incompatible *as static configura-*

[2] Note that the global symmetries, though important for explaining the appearance of the zero modes of these solitons or of their flavor quantum numbers, do not play a role in their stability.

tions. This does not mean that it is incorrect to consider a vortex which ends on a monopole. Quite the contrary! In fact, a (mesonlike) monopole-vortex-antimonopole configurations can rotate and can be dynamically stable, though as a static configuration they are not: they do not representing any nontrivial homotopy group element. After all, we believe that real-world mesons *are* quark-gluon-antiquark bound states of this sort!

It is thus perfectly sensible to consider the physics of "a vortex ending on a monopole". This notion will be made more quantitative and precise in the next section.

4 Flux Matching

Consider the configuration in which a vortex ends on a monopole (Figure 1). The vortex and monopole both represent the same minimum element of

$$\pi_2(SU(3)/(SU(2)\times U(1)/\mathbb{Z}_2)) \sim \pi_1(\frac{SU(2)\times U(1)}{\mathbb{Z}_2}) = \mathbb{Z}. \tag{4.1}$$

Therefore the total flux through an $\mathbb{R}^2$ cross-section of the vortex and the total flux through an S^2 around the monopole on which it ends, must agree (Figure 2). In the rest of this section we shall verify that this is indeed the case. This means that when the H theory is in Higgs phase the monopoles of the G/H system are indeed confined.

As a bonus, we find that the $U(1)$ charge of the nonabelian monopoles takes a fractional value with respect to the standard Dirac quantization condition, which is nicely explained by the homotopy group consideration in Ref. [23].

4.1 Monopole flux

Given the adjoint VEV

$$\langle\phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2\,v \end{pmatrix}, \tag{4.2}$$

consider a broken $SU(2)$ subgroup ("U"-spin) with generators

$$S_1 \equiv t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad S_2 \equiv t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \quad S_3 \equiv \frac{t^3+\sqrt{3}t^8}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.3}$$

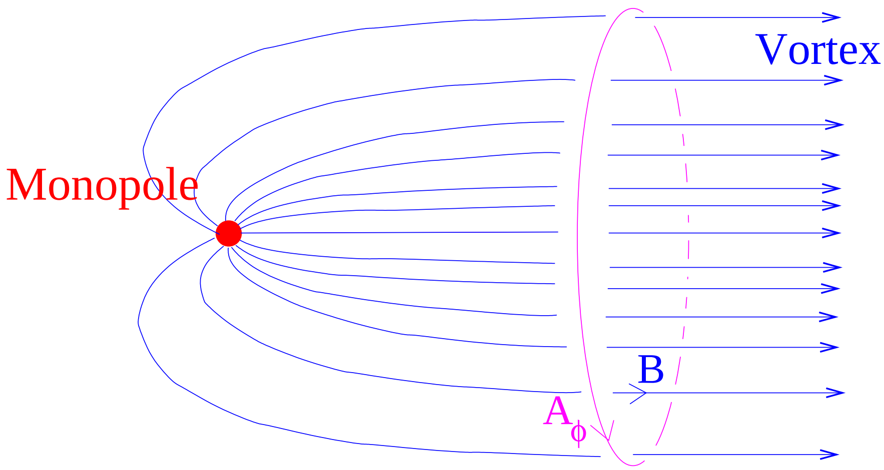

Figure 1: A single vortex ends on each monopole.

where t^k's are the Gell-Mann matrices. A nonabelian monopole transforming in the doublet of the dual of this $SU(2)$ is described by the solution [7, 15]

$$\begin{aligned} \phi(\mathbf{r}) &= \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3\, v\, \vec{S} \cdot \hat{r}\phi(r), \\ \vec{A}(\mathbf{r}) &= \vec{S} \wedge \hat{r} A(r) \end{aligned} \tag{4.4}$$

of the nonabelian Bogomolny equations (2.14), where $\phi(r) \to 1$ and $A(r) \to -\frac{1}{r}$. Another, degenerate solution can be found by making use of the "V-spin", lying in the $(2-3)$ submatrix.

The nonabelian flux is easily found to be

$$B_i = \frac{1}{2}\epsilon_{ijk}(\partial_j A_k - \partial_k A_j - i[A_j, A_k]) = \frac{r_i(\mathbf{S}\cdot\mathbf{r})}{r^4}. \tag{4.5}$$

To find the abelian flux along the unbroken $U(1)$, we project the nonabelian flux onto the direction of the adjoint scalar and compute:

$$\mathrm{Tr}\,\phi\vec{B} = \frac{3}{2}\,v\frac{\vec{r}}{r^3}. \tag{4.6}$$

Integrating this over a 2-sphere centered on the magnetic monopole and normalizing the flux such that it is independent of the absolute value of the scalar condensate, v, we find

$$F_m = \int_{S^2} d\mathbf{S}\cdot\frac{\mathrm{Tr}\,\phi\,\mathbf{B}}{\frac{1}{\sqrt{2}}(\mathrm{Tr}\,\phi\phi)^{1/2}} = 2\pi\cdot\sqrt{3}. \tag{4.7}$$

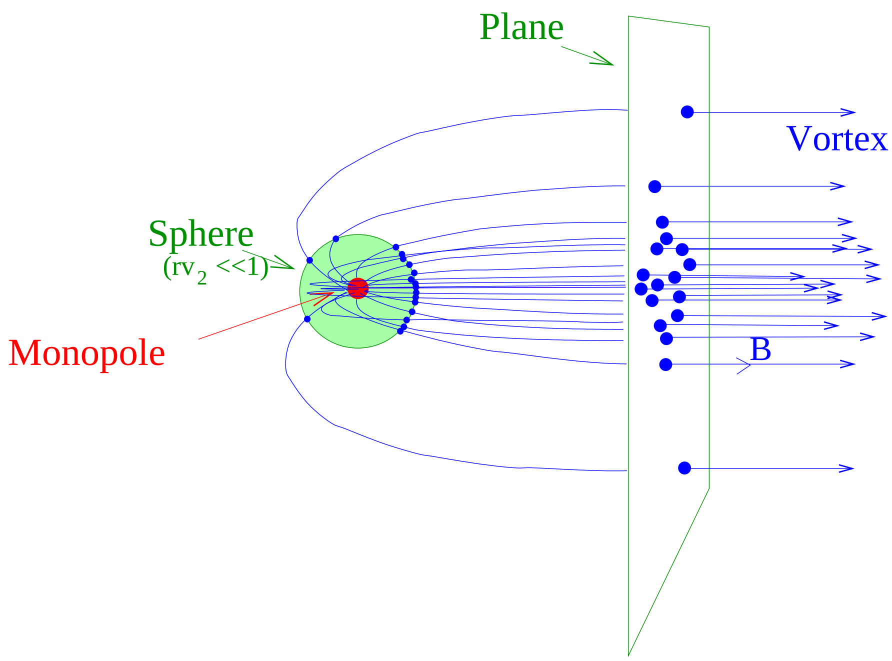

Figure 2: The total flux around the monopole, integrated over a sphere of an arbitrary radius (hence with a radius much smaller than $1/v_2$ - where it lookes like an isotropic monopole), must match the total vortex flux integrated over a plane far enough from the monopole. On this plane the vortex flux is distributed over a region much larger than $1/v_2$.

We have chosen the normalization factor of

$$\frac{1}{\sqrt{2}}(\mathrm{Tr}\,\phi\phi)^{1/2} = \frac{1}{2}v \tag{4.8}$$

since

$$\mathrm{Tr}\,\phi\,\mathbf{B} = \frac{1}{2}\,v\,B^8. \tag{4.9}$$

4.2 Vortex flux

Using the nonabelian vortex solution of [16]

$$\vec{A}_i = t^8\,A_i^8(x) + t^3\,A_i^3(x) \tag{4.10}$$

$$A_i^8(x) = -\sqrt{3}\;\epsilon_{ij}\,\frac{x_j}{r^2}\,[1 - f_8(r)] \to -\sqrt{3}\,\epsilon_{ij}\,\frac{x_j}{r^2} = -\sqrt{3}\,\frac{1}{r}\partial_i\varphi \tag{4.11}$$

$$\phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -2m \end{pmatrix} \equiv \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix} = 2\sqrt{3}\, t^8\, v \tag{4.12}$$

we find that the flux carried by a vortex is

$$\vec{B} = \nabla \wedge \vec{A}, \qquad F_v = \int_{R^2} d\mathbf{S} \cdot \frac{\mathrm{Tr}\, \phi\, \mathbf{B}}{\frac{1}{\sqrt{2}}(\mathrm{Tr}\, \phi\phi)^{1/2}} = 2\pi \cdot \sqrt{3}. \tag{4.13}$$

The monopole flux (4.7) and the vortex flux (4.13) agree precisely, and so in our theory, in contrast with that of Ref. [18], precisely one vortex ends on each monopole.

4.3 Dirac

The magnetic flux sourced by a 't Hooft-Polyakov monopole [24, 25] in a $SU(2) \to U(1)$ gauge theory is

$$F_m = \int_{S^2} d\mathbf{S} \cdot \mathbf{B} = \frac{4\pi}{g}, \qquad g_m = \frac{1}{g}. \tag{4.14}$$

Here g is the electric coupling constant, which enters the Lagrangian as

$$(\partial_\mu - i\, g \frac{\tau^a}{2} A^a_\mu)\, q \tag{4.15}$$

where q is an $SU(2)$ doublet matter field. This means that the minimum electric charge is $e_0 = \frac{g}{2}$ and so

$$g_m = \frac{1}{2\, e_0} \tag{4.16}$$

coincides with Dirac's minimum quantum of magnetic charge.

In the $SU(3) \to SU(2) \times U(1)$ theory under consideration, (4.7) means that the minimum magnetic charge is

$$g_m = \frac{\sqrt{3}}{2g}. \tag{4.17}$$

But since g enters the Lagrangian as

$$(\partial_\mu - i\, g t^a A^a_\mu) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (\partial_\mu - i\, g t^8 A^8_\mu + \ldots) \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{4.18}$$

the minimum A^8 charge is

$$e_0 = \frac{g}{2\sqrt{3}} \tag{4.19}$$

where the factor of $2\sqrt{3}$ comes from the normalization of the Gell-Mann matrix t^8. In terms of this, the magnetic charge of the doublet monopole is

$$g_m = \frac{1}{4\,e_0} \tag{4.20}$$

which is one half of the Dirac quantum.

4.4 Monopole charge and flux in an $SU(N+1)$ theory

The above analysis generalizes straightforwardly to an $SU(N+1)$ gauge theory broken to $SU(N) \times U(1)$ by the adjoint scalar VEV

$$\phi = \begin{pmatrix} v & 0 & \dots & 0 \\ 0 & v & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -Nv \end{pmatrix} = \begin{pmatrix} v \cdot \mathbf{1}_{N\times N} & \\ & -Nv \end{pmatrix}. \tag{4.21}$$

A magnetic monopole is characterized by the vector potential

$$\vec{A}(r) = \vec{S} \wedge \widehat{r}\,\frac{A(r)}{g} \tag{4.22}$$

which yields the magnetic flux

$$B_i = \frac{r_i(\mathbf{S}\cdot\mathbf{r})}{r^4} \tag{4.23}$$

where the matrices S_i

$$S_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}; \qquad S_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & \dots & i \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -i & 0 & \dots & 0 \end{pmatrix}; \tag{4.24}$$

$$S_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix} \tag{4.25}$$

generate a broken $SU(2)$ subgroup of $SU(N+1)$. In addition the adjoint Higgs field varies near the monopole as

$$\phi = \begin{pmatrix} -\frac{N-1}{2}v & 0 & \dots & 0 & 0 \\ 0 & v & 0 & \dots & 0 \\ 0 & 0 & v & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{N-1}{2}v \end{pmatrix} + (N+1)\,v\,(\vec{S}\cdot\widehat{r})\,\phi(r). \tag{4.26}$$

Tracing in the ϕ direction, normalizing by the norm of ϕ, and integrating over a sphere centered on the monopole, we find the total magnetic flux sourced by the minimal monopole

$$F_m = \int_{S^2} d\mathbf{S} \cdot \frac{\mathrm{Tr}\, \phi\, \mathbf{B}}{\frac{1}{\sqrt{2}}(\mathrm{Tr}\phi^2)^{1/2}} = 2\pi \sqrt{\frac{2(N+1)}{N}}. \tag{4.27}$$

This should be equal to $4\pi\, g_m$, and so

$$g_m = \sqrt{\frac{N+1}{2\,N}}/g. \tag{4.28}$$

On the other hand, the electric coupling of the A^0_μ field with the matter in the fundamental representation of $SU(N+1)$ is through the minimum coupling constant

$$e_0 = \frac{g}{\sqrt{2N(N+1)}}, \tag{4.29}$$

as

$$t^0 = \frac{1}{\sqrt{2N(N+1)}} \begin{pmatrix} \mathbf{1}_{N\times N} & \\ & -N \end{pmatrix}. \tag{4.30}$$

Thus the minimum magnetic charge, in terms of the unit electric charge, is

$$g_m = \frac{1}{2\,N\, e_0} \tag{4.31}$$

which is $1/N$ of the charge of Dirac's $U(1)$ monopole. This factor of N is the degree of the embedding of the fundamental group of the unbroken $U(1)$ into that of the unbroken gauge group [23].

4.5 Vortex flux in the $SU(N+1)$ theory: Flux matching

The $(1, 0, \ldots)$ - vortex solution of [16] consists of squark fields winding as

$$q^{kA} = \begin{pmatrix} e^{i\alpha}\phi_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \phi_N \end{pmatrix} \tag{4.32}$$

while the adjoint scalar field is fixed to its constant VEV. The vector potential cannot be found analytically, but instead is given in terms of the profile functions f_i which solve a particular set of differential equations

$$A^3_i(x) = -\epsilon_{ij}\, \frac{x_j}{r^2}\Big(1 - f_3\Big),$$

$$\vdots$$

$$A_i^{N^2-1}(x) = -\sqrt{\frac{2}{N(N-1)}}\epsilon_{ij}\,\frac{x_j}{r^2}\Big((1-f_{N^2-1}\Big),$$

$$A_i(x) = -\frac{1}{\tilde{e}N}\,\epsilon_{ij}\,\frac{x_j}{r^2}\Big(1-f\Big) = -\frac{1}{e}\sqrt{\frac{2(N+1)}{N}}\,\epsilon_{ij}\,\frac{x_j}{r^2} \tag{4.33}$$

where we have rescaled

$$\tilde{e} \equiv \frac{e}{\sqrt{2N(1+N)}}; \qquad \tilde{A}_i \equiv \frac{e}{\tilde{e}}A_i. \tag{4.34}$$

This may also be obtained directly directly by solving $(\nabla_i - A_i)q \to 0$, without the redefinition used in [16]. Now

$$\phi = \sqrt{2N(N+1)}\,t^0\,v$$

and so the abelian flux integrated over a cross-section of the vortex is

$$F_v = \int_{R^2} d\mathbf{S}\cdot\frac{\mathrm{Tr}\,\phi\,\mathbf{B}}{\frac{1}{\sqrt{2}}(\mathrm{Tr}\phi^2)^{1/2}} = 2\pi\sqrt{\frac{2(N+1)}{N}}, \tag{4.35}$$

in precise agreement with the monopole flux of Eq. (4.27). We see again that one vortex ends on each monopole.

4.6 Matching of nonabelian fluxes

We have seen above that the abelian parts of the monopole and vortex fluxes agree. In fact, the full nonabelian fluxes must also match precisely, if computed in the same gauges. For the monopole, the solution is asymptotically

$$\phi = \begin{pmatrix} -\frac{n-1}{2}v & 0 & \dots & 0 & 0 \\ 0 & v & 0 & \dots & 0 \\ 0 & 0 & v & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{n-1}{2}v \end{pmatrix} + (n+1)\,v\,(\vec{S}\cdot\hat{r}) \tag{4.36}$$

$$\vec{B}(r) = \frac{r_i(\mathbf{S}\cdot\mathbf{r})}{r^4}. \tag{4.37}$$

In the gauge in which ϕ is asymptotically

$$\begin{pmatrix} v & 0 & \dots & 0 \\ 0 & v & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -nv \end{pmatrix} \tag{4.38}$$

in all spatial directions, we find

$$\vec{B}(r) = \vec{S_3}\frac{r_i}{r^3} \tag{4.39}$$

and therefore the flux is given by:

$$\mathcal{F}_m = \int_{S^2} d\mathbf{S}\cdot\mathbf{B} = 4\pi S_3 = 2\pi \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}. \tag{4.40}$$

For the vortex, the solution given in (4.32),(4.33) is already in this gauge: ϕ is diagonal and constant. The vortex flux is easily found to be

$$\mathcal{F}_v = \int_{R^2} d\mathbf{S}\cdot\mathbf{B} = \int_C d\mathbf{l}\cdot\mathbf{A} = 2\pi \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix}, \tag{4.41}$$

which precisely agrees with the monopole flux.

5 Conclusion

We have thus verified that in the theory with symmetry breaking

$$SU(N+1) \xrightarrow{v_1} \frac{SU(N)\times U(1)}{\mathbb{Z}_N} \xrightarrow{v_2} 0, \qquad v_1 \gg v_2, \tag{5.1}$$

the massive monopoles representing $\pi_2(\frac{SU(N+1)}{SU(N)\times U(1)/\mathbb{Z}_N})$ are confined by the nonabelian vortices of the low-energy theory, which represent classes in $\pi_1(SU(N)\times U(1)/\mathbb{Z}_N)$. We have done so by showing that the magnetic flux of one matches exactly that of the other. As the two homotopy groups involved are isomorphic, such an agreement might appear to be automatic, but the result is by no means trivial since the monopoles

and vortices are solutions of different effective theories valid at different energy scales with different effective degrees of freedom.

As a by-product, we have checked that the $U(1)$ charge of the monopoles is indeed $\frac{1}{N}$ of the minimum Dirac quantum (for the $U(1)$ theory), a fact easily understood from the minimum closed path in the space of $SU(N) \times U(1)/\mathbb{Z}_N$ [23].

In the $SU(N+1)$ theories discussed in this paper there are no other vortices or monopoles as both the first and second homotopy groups are trivial. When the original gauge group G is not simply connected, such as in the $\frac{SU(N)}{\mathbb{Z}_N}$ or $SO(N)$ theories, there are vortices in the theory which are sourced only by external (Dirac) monopoles. These cases will be discussed elsewhere.

The most significant consequence of the flux matching discussion in this paper is the fact that the Goddard-Nuyts-Olive-Weinberg [4, 7] monopoles of the theory indeed transform as the fundamental multiplet of the dual of H, $\tilde{H} = SU(N) \times U(1)$. This follows from the fact that the vortices of the H theory in the Higgs phase are described by a continuous family of degenerate solutions (exact zero modes), parametrized by the quotient [16]

$$\mathbb{CP}^{N-1} \sim \frac{SU(N)_{C+F}}{(SU(N-1) \times U(1))_{C+F}}, \tag{5.2}$$

$(SU(N-1) \times U(1))_{C+F}$ being the invariance group of an individual vortex. As the monopoles are the sources of these vortices (nonlocal objects), it follows that the monopoles themselves transform according to the continuous, dual transformations of $\tilde{H} = SU(N) \times U(1)$ which involve nonlocal field transformations. The dual group $SU(N)$ also involves the original flavor subgroup $SU(N) \subset SU(N_f)$ in an essential manner.

The flavor symmetry group of the fundamental theory thus plays *two* crucial roles in the whole discussion. One is that through the effects of the renormalization group the fermions prevent the unbroken group H from becoming strongly coupled and breaking itself dynamically to an abelian subgroup [3]. Only in the presence of an appropriate number of massless flavors [4] the H theory remains infrared free (or conformal invariant). Otherwise, the "nonabelian monopoles" of the bosonic theory would remain simply artifacts of the semi-classical approximation [15].

[3]This is precisely what happens in pure $\mathcal{N}=2$ Yang-Mills theories or in a generic point of the moduli space of vacua.

[4]In our case, the condition is $2\,N+2 > n_f \geq 2\,N$.

Secondly, the dual group $\tilde{H}$ itself involves the original flavor group. Such a mixing of the groups of color and flavor is by now well-known (if not so well understood) as exemplified in the Seiberg duals occurring in many $\mathcal{N} = 1$ supersymmetric gauge theory models [11], and is really not surprising.

These lessons learned from the supersymmetric world, though perhaps not yet widely appreciated in the general physics community, might well be useful in understanding the phenomenon of confinement in the standard QCD.

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